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LETTER TO THE EDITOR

Anomalous crossover behaviours in the two-component deterministic percolation model

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Abstract. We have investigated the current distribution in the two-component deterministic percolation model in which the ratio h of poor to good conductance is regarded as a small parameter. It is found that the minimum current $I_{\min}(h)$ scales anomalously with h: $I_{\min}(h)/I_{\min}(1)=\exp(-\text{constant}(\ln h)^2)H(hL^{\phi})$, where L is the size of the network, H a function describing the crossover from fractal to homogeneous behaviours and ϕ the crossover exponent. The exponential prefactor is quite similar to the behaviour of left-sided multifractality in diffusion limited aggregations. It is found that ϕ coincides with the crossover exponent for all multifractal moments of the current distribution.

Recently there has been an increased interest in the study of the current distribution in a random resistor network (RRN). It is proposed that different moments of the current distribution scale with an infinite number of independent exponents [1]. The problem has been extended to the two-component RRN which is composed of good conductors with impedance X at a concentration p and poor conductors with impedance Y at a concentration $1-p$. The impedance ratio $h=X/Y$ is regarded as a small parameter and one focuses on how the moments scale within fractal ($h=0$) and homogeneous ($h=1$) regions. For such a network, the question of whether there are an infinite number of exponents describing the crossover from fractal to homogeneous behaviours has attracted current interest [2–4]. However, the study of the scaling of the negative moments is lacking.

The negative moments of the current distribution are dominated by the minimum current of the network, which is in general very difficult to determine accurately in numerical simulations. In this letter, we study a deterministic percolation model (DPM) [5, 6] which is constructed iterately so that one can obtain the complete set of currents exactly. It is quite surprising to find that the minimum current $I_{\min}(h)$ scales anomalously with h:

$$I_{\min}(h)/I_{\min}(1)=\exp(-\text{constant}(\ln h)^2)H(hL^{\phi}) \tag{1}$$

where L is the size of the network, H a function describing the crossover behaviour and ϕ the crossover exponent. The exponential prefactor is quite similar to the behaviour of left-sided multifractality in diffusion limited aggregations [7] in that the minimum growth probability scales with the size L of the aggregate exponentially as $\exp[-c(\ln L)^2]$ but we should emphasize here that the analogue is in a completely different context. It is also interesting to find that ϕ coincides with the crossover exponent for all multifractal moments of the current distribution. The present $\exp[-c(\ln h)^2]$ behaviour will be confirmed in real two-component RRNs [8].

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The deterministic percolation model (DPM) is an extension of the fractal lattice which was originally proposed by Kirkpatrick [5] to model the percolation backbone. Here we follow Clerc *et al* [6] to extend the constructions to the two-component RRN. Starting with a filled square, one divides it into four equal squares and replaces the right upper quadrant by an empty square to obtain the first generation. The lateral size is increased by a factor of two. The second generation is obtained from the first generation by replacing each square by two types of generators. The generator for a filled square is exactly the same as the first generation but the generator for an empty square is complementary to that of the filled square, i.e. the upper right quadrant is filled while the rest are empty. The process is repeated *ad infinitum* to obtain the DPM.

It is interesting to find that the DPM consists of finite clusters and dangling bonds as well as a connected cluster for conduction, which mimics a real percolating network. Moreover, although the model thus constructed has a filling factor $f=\frac{3}{4}$ we adopt in subsequent studies an arbitrary filling factor $f(\frac{1}{2}<f<1)$. An equivalent circuit model [6] is shown in figure 1(*a*) in that the good conducting bond at the right is a parallel combination of $(4f-2)$ good conductors while that at the left has $4(1-f)$ poor conductors. Similar notations are used in figure 1(*b*) to model the poor conducting bonds.

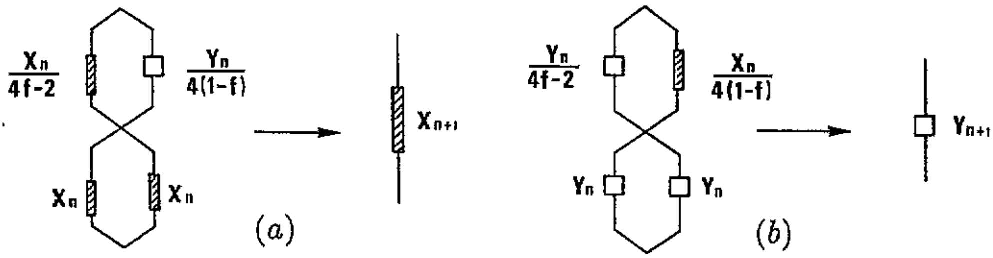

Figure 1. The renormalization process of DPM: (*a*) good conductor → good conductor, (*b*) poor conductor → poor conductor.

For completeness, we summarize the salient geometrical properties of the DPM as follows. The total number of circuit elements N_n at the nth generation is 4^n with a lateral size $L=2^n$. The fractal dimension is therefore two. The number of good conductors is

$$N_c=\tfrac{1}{2}N_n[1+(2f-1)^n]$$

while the number of poor conductors is

$$N_I=\tfrac{1}{2}N_n[1-(2f-1)^n].$$

As the size increases, these numbers converge to $\frac{1}{2}N_n$; the fraction of good conductors approaches the percolation threshold of a two-dimensional bond percolation model ($p_c=\frac{1}{2}$).

Here for the purpose of establishing notations, we follow Clerc *et al* [6] to discuss briefly the scaling properties of the impedance of the DPM. From figure 1, the impedance of the good conductor X_n and that of the poor conductor Y_n at the nth generation are related by the following recursion relations

$$X_n=\tfrac{1}{2}X_{n-1}+\frac{X_{n-1}Y_{n-1}}{4(1-f)X_{n-1}+(4f-2)Y_{n-1}} \tag{2a}$$

$$Y_n=\tfrac{1}{2}Y_{n-1}+\frac{Y_{n-1}X_{n-1}}{4(1-f)Y_{n-1}+(4f-2)X_{n-1}}. \tag{2b}$$

Note that the roles of X_n and Y_n are interchanged in equations (2a) and (2b). If we denote $h_n = X_n/Y_n$, we arrive at a recursion relation for h_n,

$$h_n = S(h_{n-1}) \tag{3}$$

where $S(h)$ is a one-parameter iterated map,

$$S(h) = h\frac{[(1-f)h+f][(2f-1)h+2(1-f)]}{[fh+(1-f)][2(1-f)h+(2f-1)]}. \tag{4}$$

The solutions of $h = S(h)$ give the fixed points for the iterated map, here we have four fixed points at $h=-1, 0, 1, \infty$. Stability analysis shows that $h=1$ is the only stable fixed point while the others are unstable. In the vicinity of the fixed point, we defined the fixed point multiplier

$$\lambda_{\text{fp}} = \left.\frac{dS(h)}{dh}\right|_{\text{fp}}. \tag{5}$$

At $h=-1, \lambda_{-1} = (8f^2-12f+5)/(2f-1)(4f-3)$; $h=0, \lambda_0 = 2f/(2f-1)$; $h=1, \lambda_1 = 2f-1$; $h=\infty, \lambda_\infty = 2f/(2f-1)$. Since λ_1 is less than unity, we conclude that $h=1$ is a stable fixed point. The unstable fixed point $h=0$ corresponds to the interesting cases of the random resistor network limit in which the poor conductor has no finite conductance and the random superconducting network limit in which the good conductor has an infinite conductance. We are interested in the scaling region with a small but finite initial value h_0 (in the vicinity of the $h=0$ unstable fixed point) in a finite network so that the subsequent flow to the $h=1$ fixed point leads to a crossover from the fractal to the homogeneous behaviour.

Let us investigate the scaling behaviour of impedance at $h=0$. We define a function $\varphi(h)$ such that

$$X_n = X_{n-1}\varphi(h_{n-1}) = X_0\varphi(h_{n-1})\ldots\varphi(h_2)\varphi(h_1)\varphi(h_0) \tag{6}$$

where

$$\varphi(h) = \frac{(1-f)h+f}{2(1-f)h+(2f-1)}. \tag{7}$$

For $h_0=0$, it follows that $h_1 = h_2 = \ldots = h_{n-1} = 0$ and $\varphi(h_n) = \varphi(0)$ for all n. Hence, $X_n = X_0[\varphi(0)]^n$. If we write $X_n = X_0 L^{t/\nu}$, where $L=2^n$ being the size of the network, we identify the conductivity exponent $t/\nu = \ln\varphi(0)/\ln 2$. Due to the recursive nature of the DPM, we are able to derive a homogeneous relation

$$X_n = X_0 L^{t/\nu} H_t(h_0 L^{\phi}) \tag{8}$$

where $\phi = (s+t)/\nu$ is the crossover exponent and $s/\nu = [\ln\lambda_0 - \ln\varphi(0)]/\ln 2$ is the superconducting exponent [6]. We have computed X_n numerically using equation (6) and plotted $X_n L^{t/\nu}$ against $h_0 L^{(s+t)/\nu}$ in a log–log plot with various initial values of h_0 and network sizes L. The data all lie on a universal curve, thus confirming the homogeneous relation equation (8).

Next we find that in fact all currents scale with the same crossover exponent $\phi = (s+t)/\nu$. From figure 2 and by using elementary circuit equations, one can determine the complete set of currents in the good conductor

$$\begin{aligned} I_{n1} &= (4f-2)(\varphi(h_n)-\tfrac{1}{2})I_{n+1} \\ I_{n2} &= 4(1-f)h_n(\varphi(h_n)-\tfrac{1}{2})I_{n+1} \\ I_{n3} &= I_{n4} = \tfrac{1}{2}I_{n+1}. \end{aligned} \tag{9}$$

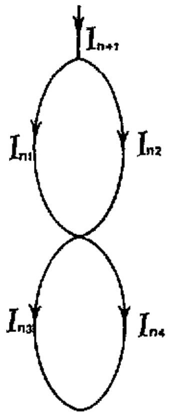

Figure 2. The four branches of the current flow in an arbitrary generation.

The maximum current is located at the first bond of the good conductors (see figure 1(a) and figure 2) at any generation. The maximum current at the nth generation is given by

$$I_{\max}^{(n)} = (4f-2)^n(\varphi(h_{n-1})-\tfrac{1}{2})(\varphi(h_{n-2})-\tfrac{1}{2})\ldots(\varphi(h_0)-\tfrac{1}{2})I_n.$$

Since $(\varphi(0)-\tfrac{1}{2})=(4f-2)^{-1}$, one finds

$$I_{\max}^{(n)} = \prod_{m=0}^{n-1} \frac{\varphi(h_m)-\frac{1}{2}}{\varphi(0)-\frac{1}{2}} = H_{\max}(x) \tag{10}$$

where $x=\lambda_0^n h_0 = h_0 L^{\phi}$ is the relevant scaling variable. We have plotted the $I_{\max}^{(n)}$ against $h_0 L^{\phi}$, with different h_0 and size L, and they all lie on a universal curve $H_{\max}(x)$.

The aim of the present study is to extract the minimum current and to examine its anomalous crossover behaviour. The minimum current of the network is located at the first bond of the poor conductors (see figure 1(b) and figure 2) at any generation. Similar to equation (9), we can easily obtain the complete set of current distribution in the poor conductor,

$$\begin{aligned} I_{n1} &= (4f-2)(\varphi(h_n)h_n/h_{n+1}-\tfrac{1}{2})I_{n+1} \\ I_{n2} &= 4(1-f)h_n^{-1}(\varphi(h_n)h_n/h_{n+1}-\tfrac{1}{2})I_{n+1} \\ I_{n3} &= I_{n4} = I_{n+1}/2. \end{aligned} \tag{11}$$

We can write the minimum current at the nth generation as

$$I_{\min}^{(n)} = (4f-2)^{n-1} \prod_{m=0}^{n-1} (\varphi(h_{m-1})h_{m-1}/h_m - \tfrac{1}{2})I_{n-1}$$

with

$$I_{n-1} = 4(1-f)h_{n-1}(\varphi(h_{n-1})-\tfrac{1}{2})I_n.$$

Let us define a new function $F_0(x)$ such that

$$F_0(x) = \lim_{m\to\infty} S^m(x/\lambda_0^m). \tag{12}$$

Shifting x to $\lambda_0 x$, one can show that $F_0(\lambda_0 x)=S(F_0(x))$. Hence we obtain an asymptotic expansion for $F_0(x)$ at small h:

$$F_0(x)=x-\left(\frac{4f^2-5f+2}{2f(1-f)}\right)x^2+\ldots. \tag{13}$$

We arrive at

$$I_{\min}^{(n)}=(4f-2)^{n-1}4(1-f)h_0(\varphi[F_0(x/\lambda_0)]-\tfrac{1}{2}) \times \prod_{m=1}^{n-1}\left(\varphi[F(x/\lambda_0^{n-m+1})]-\frac{F_0(x/\lambda_0^{n-m})}{2F_0(x/\lambda_0^{n-m+1})}\right) \tag{14}$$

again $x=\lambda_0^n h_0=h_0 L^{\phi}$ is the relevant scaling variable. One can readily see that if we let $h_m=1$ and thus $\varphi(h_m)=\varphi(1)=1$ for all m, then

$$I_{\min}^{(n+1)}(1)=(4f-2)^n 4(1-f)(1/2)^{n+1}.$$

We can therefore normalize $I_{\min}^{(n+1)}$ by $I_{\min}^{(n+1)}(1)$

$$\frac{I_{\min}^{(n+1)}(h)}{I_{\min}^{(n+1)}(1)}=h_0 2(\varphi(h_n)-\tfrac{1}{2})\prod_{m=1}^{n}2\left(\varphi[F(x/\lambda_0^{n-m+1})]-\frac{F_0(x/\lambda_0^{n-m})}{2F_0(x/\lambda_0^{n-m+1})}\right). \tag{15}$$

In order to extract the scaling and the crossover properties of the minimum current from equation (15), let us consider a small initial value of h_0 and a large generation n. Most of the terms in the product involve small values of arguments of x/λ_0^{n-m} so that their values are small except for the last few terms which are close to unity. There exists a typical value of $n=n^*$ beyond which $x'=x/\lambda_0^{n-n^*}\approx 1$ and that the product of the terms for $n>n^*$ is essentially unity. We can therefore consider the product from $m=1$ to n^*. The value of n^* which depends on the initial value of h_0 can be estimated as follows. With $h_{n^*}\approx 1$ or $\lambda_0^{n^*}h_0\approx 1$, we have $n^*\approx -\ln h_0/\ln\lambda_0$. For small $h\to 0$, then $\varphi(h)\to\varphi(0)$, $h_m/h_{m-1}\to\lambda_0$. Since $\lambda_0=2\varphi(0)$, $I_{\min}^{(n+1)}(0)\to 0$, we have to use the first-order expansions of $\varphi(h)$ and $F_0(x)$ near $h\to 0$ and $x\to 0$ respectively.

$$\varphi(h)=\frac{f}{2f-1}\left(1-\frac{(1-f)h}{f(2f-1)}+\ldots\right)$$

and

$$F_0(x)=x-\left(\frac{4f^2-5f+2}{2f(1-f)}\right)x^2+\ldots.$$

We obtain the minimum current

$$I_{\min}^{(n+1)}(h_0)/I_{\min}^{(n+1)}(1)=\lambda_0^{-n^*(n^*+1)/2}H(\lambda_0^n h_0)$$

which is dominated by the factor $\lambda_0^{-n^*(n^*+1)/2}$. Putting in the value of $n^*\approx -\ln h_0/\ln\lambda_0$, we arrive at the desired result [see equation (1)]

$$I_{\min}^{(n+1)}(h_0)/I_{\min}^{(n+1)}(1)=\exp(-(\ln h_0)^2/2\ln\lambda_0)H(\lambda_0^n h_0). \tag{16}$$

In figure 3, we plot $I_{\min}^{(n+1)}(h_0)/I_{\min}^{(n+1)}(1)$ as a function of $h_0 L^{\phi}$ in a log-log plot, we can see that as n increases the curve crosses over from a rapidly decreasing region towards a constant. The crossover occurs clearly at $n\approx n^*(h_0)\approx -\ln h_0/\ln\lambda_0$. In figure 4, we plot the rescaled minimum current $(I_{\min}^{(n+1)}(h_0)/I_{\min}^{(n+1)}(1))\exp(-(\ln h_0)^2/2\ln\lambda_0)$ against $h_0 L^{\phi}$, the collapse of all data on a universal curve is evident. In fact, the dependence of the minimum current on h is very similar to that of the minimum growth

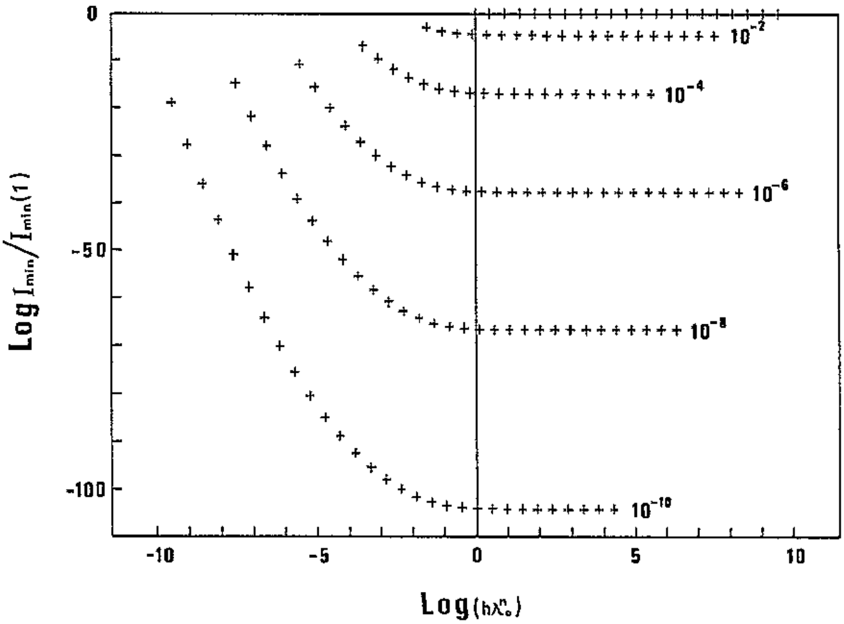

Figure 3. Log-log plot of the normalized minimum current $I_{min}(h)/I_{min}(1)$ as a function of h_0L^{ϕ} for $h_0=1$, 10^{-2}, 10^{-4} and n from 1 to 20 and for $h_0=10^{-6}$, 10^{-8}, 10^{-10} and n from 1 to 30. The filling factor is $f=\frac{3}{4}$.

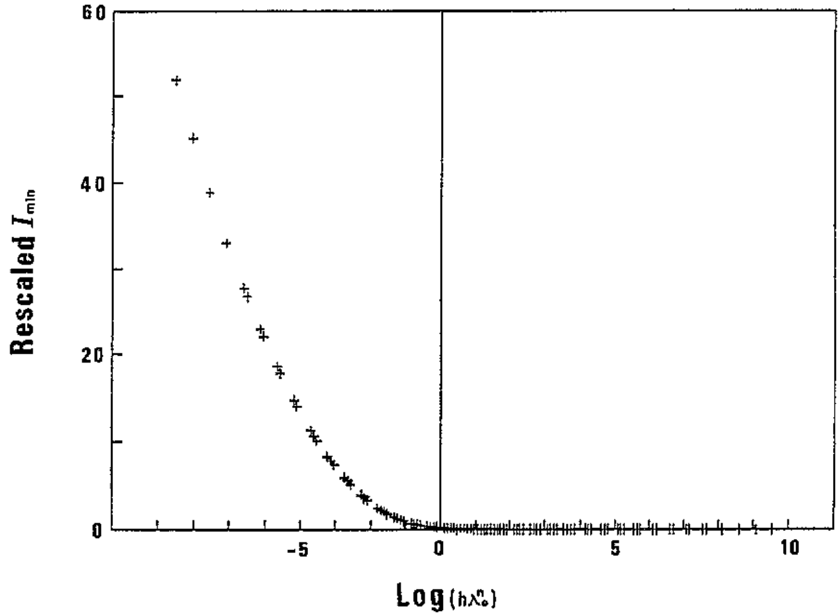

Figure 4. Log-log plot of the rescaled minimum current $(I_{min}(h)/I_{min}(1))(\lambda_0^{-n^*(n^*+1)/2})$ as a function of h_0L^{ϕ}. The same set of data is used as in figure 3.

probability of a diffusion limited aggregation on the size, both exhibiting decay faster than any power law. However, the present study is in a completely different context from the behaviour of left-sided multifractality in diffusion limited aggregations.

A few comments on our model are in order. First of all, let us study the generality of the results. We have performed similar calculations in the two-component diamond lattice [9], which is the exact dual lattice of DPM. We also performed numerical simulations on the two-dimensional random resistor networks. To this end, we should remark that the minimum current is extremely difficult to determine accurately in numerical simulations. We instead determine the most probable current which can be shown to scale in the same way as the minimum current does (see below). We obtained the same $\exp[-c(\ln h)^2]$ behaviour [8]. Perhaps one might suspect that the index '2' could be the fractal dimension of the underlying lattice. We did similar calculations in the de Arcangelis-Redner-Coniglio (ARC) hierarchical lattice [1]. The same $\exp[-c(\ln h)^2]$ is obtained although the ARC lattice has a non-integral dimension.

Secondly, we develop the multifractal aspect of the model, i.e. we compute the left side of the $f(\alpha)$ spectrum. We consider the partition function $\Sigma_i I_i^{2q} \approx L^{-\tau(q)}$ according to multifractal analysis [10]. For $h=1$, we find $\tau(q)=2-q$, which indicates a constant-gap scaling. The $f(\alpha)$ spectrum is a single point at $\alpha=1$ and $f=2$. For $h=0$, we recover the deterministic fractal lattice of [5]. When $h\to 0$, the maximum current $I_{\max}\to 1$. For $q\geqslant 0$ and for sufficiently large size $L=2^n$, the partition function scales as

$$\sum_i I_i^{2q} \approx (1+2^{-(2q-1)})^n.$$

We obtain $\tau(q)=\ln[1+2^{-(2q-1)}]/\ln 2$. Tremblay *et al* [11] did Monte Carlo simulations for integral values of $q=1$, 2 and 3 only. We find $\tau(1)=-0.585$, $\tau(2)=-0.167$ and $\tau(3)=-0.044$, *in qualitative agreements with the results* $\tau(1)=-0.98$, $\tau(2)=-0.82$ *and* $\tau(3)=-0.77$ of Tremblay *et al* [11]; in particular the inequality $|\tau(q)|>|\tau(q+1)|$ is strictly obeyed. One should not be too surprised by such a discrepancy because deterministic fractal models can only capture qualitatively the scaling behaviour of percolation clusters.

We have presented analytic calculations for the maximum and minimum currents only. It is also interesting to examine the size and conductance-ratio dependence of the current distribution. Due to the iterative nature of the model, we are able to find exact recursion relations for the current distribution. Here we briefly summarize our results and details will be presented elsewhere [8]. For extremely small h such that $hL^{\phi}\ll 1$, we find that $D(\alpha)\approx\exp(-A[\alpha-B(\log L)^2]/(\log L)^3]$, where $D(\alpha)\,\mathrm{d}\alpha$ is the number of currents with $\alpha<-\log I_i<\alpha+\mathrm{d}\alpha$; I_i is the current in bond i and A, B are constants. The current distribution is well approximated by a Gaussian with the mean varying with size as $(\log L)^2$ while the variance as $(\log L)^3$. As a result, both the most probable current and the minimum current scale with L as $\exp[-c(\log L)^2]$. In the opposite limit $hL^{\phi}\gg 1$, the current distribution reduces to the trivial form of a one-component lattice, i.e. it remains narrow.

In conclusion, we have investigated the crossover behaviour of the minimum current in the deterministic percolation model. It is found that the minimum current $I_{\min}(h)$ scales anomalously with h:

$$I_{\min}(h)/I_{\min}(1)=\exp(-\text{constant}(\ln h)^2)H(hL^{\phi}).$$

The exponential prefactor is quite similar but in a completely different context from the behaviour of left-sided multifractality in diffusion-limited aggregations. As we find that all currents scale with the same crossover exponent, we conclude that all multifractal moments of the current distribution scale with a single crossover exponent $\phi = (s+t)/\nu$. We suggest numerical simulations in real percolation system be done to check our scaling predictions [8].

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